Sir George Stokes and the concept of uniform convergence. By

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[*Received* 1 Jan. 1918. *Read* 4 Feb. 1918.]

1. The discovery of the notion of uniform convergence is generally and rightly attributed to Weierstrass, Stokes, and Seidel. The idea is present implicitly in Abel's proof of his celebrated theorem on the continuity of power series; but the three mathematicians mentioned were the first to recognise it explicitly and formulate it in general terms*. Their work was quite independent, and it would be generally agreed that the debt which mathematics owes to each of them is in no way diminished by any anticipation on the part of the others. Each, as it happens, has some special claim to recognition. Weierstrass's discovery was the earliest, and he alone fully realised its far-reaching importance as one of the fundamental ideas of analysis. Stokes has the actual priority of publication; and Seidel's work is but a year later and, while narrower in its scope than that of Stokes, is even sharper and clearer.

My object in writing this note is to call attention to and, so far as I can, explain two puzzling features in the justly famous memoir† in which Stokes announces his discovery. The memoir is remarkable in many respects, containing a general discussion of the possible modes of convergence, both of series and of integrals, far in advance of the current ideas of the time. It contains also two serious mistakes, mistakes which seem at first sight almost inexplicable on the part of a mathematician of so much originality and penetration.

The first mistake is one of omission. It does not seem to have occurred to Stokes that his discovery had any bearing whatever on the question of term by term integration of an infinite series. The same criticism, it is true, may be made of Seidel's paper. But Seidel is merely silent on the subject. Stokes, on the other hand, quotes the false theorem that a convergent series may always be integrated term by term, and refers, apparently with approval, to the erroneous proof offered by Cauchy and Moigno‡.

Of this there is, I think, a fairly simple and indeed a double

* The idea was rediscovered by Cauchy, five or six years after the publication of the work of Stokes and Seidel. See Pringsheim, 'Grundlagen der allgemeinen Funktionenlehre', *Encykl. der Math. Wiss.*, II A 1, § 17, p. 35.

† 'On the critical values of the sums of periodic series', *Trans. Camb. Phil. Soc.*, vol. 8, 1847, pp. 533–583 (*Mathematical and physical papers*, vol. 1, pp. 236–313).

‡ See p. 242 of Stokes's memoir (as printed in the collected papers).

explanation. In the first place it must be remembered that Stokes was primarily a mathematical physicist. He was also a most acute pure mathematician; but he approached pure mathematics in the spirit in which a physicist approaches natural phenomena, not looking for difficulties, but trying to explain those which forced themselves upon his attention. The difficulties connected with continuity and discontinuity are of this character. The theorem that a convergent series of continuous functions has necessarily a continuous sum is one whose falsity is open and aggressive: examples to the contrary obtrude themselves on analyst and physicist alike. The falsity of this theorem Stokes therefore observed and corrected. The falsity of the corresponding theorem concerning integration lies somewhat deeper. It is easy enough, when one's attention has been called to it, to see that the proof of Cauchy and Moigno is invalid. But there are no particularly obvious examples to the contrary: simple and natural examples are indeed somewhat difficult to construct*. And Stokes, his suspicions never having been excited, seems to have accepted the false theorem without examination or reflection.

This is half the explanation. The second half, I think, lies in the distinctions between different modes of uniform convergence which I shall consider in a moment.

Stokes's second mistake is more obvious and striking. He proves, quite accurately, that uniform convergence implies continuity†. He then enunciates and offers a proof‡ of the converse theorem, which is false. The error is not one merely of haste or inattention. The argument is as explicit and as clearly stated in one case as in the other; and, up to the last sentence, it is perfectly correct. He proves that continuity involves *something*, and then states, without further argument, that this something is what he has just defined as uniform convergence. It is merely this last statement that is false.

Stokes's mistake seems at first sight so palpable that I was for some time quite at a loss to imagine how he could have made it. A closer examination of his memoir, and a comparison of his work with other work of a very much later date, has made the lapse a good deal more intelligible to me; and my attempts to understand it have led me to a number of remarks which, although they contain very little that is really novel, are, I think, of some historical and intrinsic interest.

2. There are no less than *seven* different senses, all important, in which a series may be said to be uniformly convergent.

* See Bromwich, *Infinite series*, pp. 116–118; Hardy, 'Notes on some points in the integral calculus', XL, *Messenger of Mathematics*, vol. 44, 1915, pp. 145–149.

† p. 282. I use 'uniform' instead of Stokes's 'not infinitely slow'.

‡ p. 283.

I shall write the series in the form

$$\sum_1^\infty u_n(x);$$

and I shall suppose, for simplicity, that every term of the series is continuous, and the series convergent, for every x of the interval $a \leqslant x \leqslant b$. I shall denote the sum of the series by $s(x)$; and I shall write

$$s_n(x) = u_1(x) + u_2(x) + \dots + u_n(x), \quad s(x) = s_n(x) + r_n(x).$$

The fundamental inequality in all my definitions will be of the type

$$|r_n(x)| \leqslant \epsilon \quad \text{(A)}.$$

I shall refer to this inequality simply as (A).

When we define uniform convergence, in one sense or another, we have to choose various numbers in a definite logical order, those which are chosen later being, in general, functions of those which are chosen before. I shall write each number in a form in which all the arguments of which it is a function appear explicitly: thus $n_0(\xi, \epsilon)$ is a function of ξ and ϵ, $n_0(\epsilon)$ one of ϵ alone.

It will sometimes happen that one of the later numbers depends upon several earlier numbers *already connected by functional relations*, so that it is really a function of a selection of these numbers only. Thus δ may have been determined as a function of ϵ; and n_0 may have to be determined as a function of ξ, ϵ, and δ, so that it is in reality a function of ξ and ϵ only. I shall express this by writing

$$n_0 = n_0(\xi, \epsilon, \delta) = n_0(\xi, \epsilon);$$

and I shall use a similar notation in other cases of the same kind.

3. The first three senses of uniform convergence are as follows.

A 1: Uniform convergence throughout an interval. *The series is said to be uniformly convergent throughout the interval* (a, b) *if to every positive* ϵ *corresponds an* $n_0(\epsilon)$ *such that* (A) *is true for* $n \geqslant n_0(\epsilon)$ *and* $a \leqslant x \leqslant b$.

This is the ordinary or 'classical', and most important, sense, the sense in which uniform convergence is defined in every treatise on the theory of series.

A 2: Uniform convergence in the neighbourhood of a point. *The series is said to be uniformly convergent in the neighbourhood of the point* ξ *of the interval* (a, b) *if an interval* $(\xi - \delta(\xi), \xi + \delta(\xi))$* *can be found throughout which it is uniformly convergent; that is to say if a positive* $\delta(\xi)$ *exists such that* (A) *is true for every positive* ϵ, *for* $n \geqslant n_0(\xi, \delta, \epsilon) = n_0(\xi, \epsilon)$, *and for* $\xi - \delta(\xi) \leqslant x \leqslant \xi + \delta(\xi)$.

* A trivial change is of course required in the definition if $\xi = a$ or $\xi = b$. The same point naturally arises in the later definitions.

A 3: Uniform convergence at a point. *The series is said to be uniformly convergent at the point $x = \xi$ (or for $x = \xi$) if to every positive ϵ correspond a positive $\delta(\xi, \epsilon)$ and an $n_0(\xi, \epsilon, \delta) = n_0(\xi, \epsilon)$ such that* (A) *is true for $n \geqslant n_0(\xi, \epsilon)$ and for $\xi - \delta(\xi, \epsilon) \leqslant x \leqslant \xi + \delta(\xi, \epsilon)$.*

4. Before proceeding further it will be well to make a few remarks concerning these definitions and their relations to one another.

The idea of uniform convergence *in the neighbourhood of a particular point* (Definition **A 2**) is substantially that defined by Seidel in 1848*. It is clear, however, that definitions **A 1** and **A 2** were both familiar to Weierstrass as early as 1841 or 1842†. It is obvious that a series uniformly convergent throughout an interval is uniformly convergent in the neighbourhood of every point of the interval. The converse theorem is important and by no means obvious, and was first proved by Weierstrass‡ in a memoir published in 1880. This theorem would now be proved by a simple application of the 'Heine-Borel Theorem', and is a particular case of a theorem which will be referred to in a moment.

Definition **A 3** appears first, in the form in which I state it, in a paper of W. H. Young published in 1903§; but the idea is present in an earlier paper of Osgood‖. The essential difference between definitions **A 2** and **A 3** is that in the latter δ is chosen *after* ϵ and is a function of ξ and ϵ, while in the former it is chosen before ϵ and is a function of ξ alone. In each case n_0 is a function of two independent variables, ξ and ϵ. It is plain that uniform convergence in the neighbourhood of ξ involves uniform convergence at ξ, and at (and indeed in the neighbourhood of) all points sufficiently near to ξ. But uniform convergence at ξ does not involve uniform convergence in the neighbourhood of ξ.

It is important, however, to observe that *uniform convergence at every point of an interval involves uniform convergence throughout the interval.* This important theorem is proved very simply by

* 'Note über eine Eigenschaft der Reihen, welche discontinuirliche Functionen darstellen', *Münchener Abhandlungen*, vol. 7, 1848, pp. 381–394. This memoir has been reprinted in Ostwald's *Klassiker der exakten Wissenschaften*, no. 116. The reference there given to vol. 5, 1847, is incorrect.

† For detailed references bearing on this and similar historical points, see Pringsheim's article already quoted.

‡ See the memoir 'Zur Functionenlehre' (*Abhandlungen aus der Funktionenlehre*, pp. 69–104 (pp. 71–72)).

§ 'On non-uniform convergence and term-by-term integration of series', *Proc. London Math. Soc.*, ser. 2, vol. 1, pp. 89–102.

‖ 'Non-uniform convergence and the integration of series', *American Journal of Math.*, vol. 19, 1897, pp. 155–190. See Prof. Young's remarks on this point at the beginning of his later paper 'On uniform and non-uniform convergence of a series of continuous functions and the distinction of right and left', *Proc. London Math. Soc.*, ser. 2, vol. 6, 1907, pp. 29–51.

Young, in his paper already quoted, by means of the Heine-Borel Theorem*; and it plainly includes, as a particular case, Weierstrass's theorem referred to above.

5. It seems to me that the definition given by Stokes is not any one of **A 1**, **A 2**, **A 3**; and that, if we are to understand him rightly, we must consider another parallel group of definitions. These definitions differ from those given above in that (A) is supposed to be satisfied, not for *all* sufficiently large values of n, but only for *an infinity of* values.

B 1: Quasi-uniform convergence throughout an interval. *The series is said to be quasi-uniformly convergent throughout* (a, b) *if to every positive* ϵ *and every* N *corresponds an* $n_0(\epsilon, N)$ *greater than* N *and such that* (A) *is true for* $n = n_0(\epsilon, N)$ *and* $a \leqslant x \leqslant b$.

B 2: Quasi-uniform convergence in the neighbourhood of a point. *The series is said to be quasi-uniformly convergent in the neighbourhood of* ξ *if an interval* $(\xi - \delta(\xi), \xi + \delta(\xi))$ *can be found throughout which it is quasi-uniformly convergent; i.e., if a positive* $\delta(\xi)$ *exists such that* (A) *is true for every positive* ϵ, *every* N, *an* $n_0(\xi, \delta, \epsilon, N) = n_0(\xi, \epsilon, N)$ *greater than* N, *and* $\xi - \delta(\xi) \leqslant x \leqslant \xi + \delta(\xi)$.

B 3: Quasi-uniform convergence at a point. *The series is said to be quasi-uniformly convergent for* $x = \xi$ *if to every positive* ϵ *and every* N *correspond a positive* $\delta(\xi, \epsilon, N)$ *and an*

$$n_0(\xi, \epsilon, \delta, N) = n_0(\xi, \epsilon, N),$$

greater than N, *such that* (A) *is true for* $n = n_0(\xi, \epsilon, N)$ *and for* $\xi - \delta(\xi, \epsilon, N) \leqslant x \leqslant \xi + \delta(\xi, \epsilon, N)$.

Definition **B 1** is to be attributed to Dini or to Darboux†. Another form of it has been given by Hobson‡. As Arzelà and Hobson§ have pointed out, a series is quasi-uniformly convergent throughout an interval if, and only if, it can be made uniformly convergent by an appropriate bracketing of its terms.

Definition **B 2** is for us at the moment of peculiar interest, for (as I shall show in a moment) it is really *this* definition that is given by Stokes.

Definition **B 3** is also of great interest, both in itself and in

* Choose ϵ and determine $\delta(\xi, \epsilon)$ and $n_0(\xi, \epsilon)$, as in definition **A 3**, for every ξ of the interval. Every point of (a, b) is included in an interval $(\xi - \delta, \xi + \delta)$. By the Heine-Borel Theorem, every point of (a, b) is included in one or other of a finite sub-set of these intervals. If $N(\epsilon)$ is the largest of the n_0's corresponding to each of the intervals of this finite sub-set, then (A) is true for $n \geqslant N$ and $a \leqslant x \leqslant b$.

This is the essence of the proof, though, like all proofs of the same character, it requires a somewhat more careful statement if all appearance of dependence upon Zermelo's *Auswahlsprinzip* is to be avoided.

† See Pringsheim, *l. c.*

‡ 'On modes of convergence of an infinite series of functions of a real variable', *Proc. London Math. Soc.*, ser. 2, vol. 1, 1903, pp. 373–387. Hobson (following Dini) uses the expression 'simply uniformly'.

§ *L. c.*, p. 375.

relation to Stokes's memoir. For *the necessary and sufficient condition that $s(x)$ should be continuous for $x = \xi$ is that the series should be quasi-uniformly convergent for $x = \xi$.* This theorem is in substance due to Dini*. I give the proof, as it is essential for the criticism of Stokes's memoir.

(1) *The condition is sufficient.* For

$$|s(x) - s(\xi)| \leqslant |s_n(x) - s_n(\xi)| + |r_n(x)| + |r_n(\xi)|.$$

Choose ϵ, N, $\delta(\xi, \epsilon, N)$, and $n = n_0(\xi, \epsilon, N)$ as in definition **B 3**. Then $|r_n(x)| < \epsilon$ for $\xi - \delta \leqslant x \leqslant \xi + \delta$. Now that n is fixed we can choose δ_1 less than δ and such that $|s_n(x) - s_n(\xi)| < \epsilon$ for $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$. And thus

$$|s(x) - s(\xi)| < 3\epsilon$$

for $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$, so that $s(x)$ is continuous for $x = \xi$.

It is plain that this argument proves, *a fortiori*, that **A 2**, **A 3**, and **B 2** all furnish sufficient conditions for continuity at a point, and **A 1** and **B 1** sufficient conditions for continuity throughout an interval.

(2) *The condition is necessary.* For

$$|r_n(x)| \leqslant |s(x) - s(\xi)| + |r_n(\xi)| + |s_n(x) - s_n(\xi)|.$$

Suppose that ϵ and N are given. Then we can choose $\delta(\xi, \epsilon)$ so that $|s(x) - s(\xi)| < \epsilon$ for $\xi - \delta \leqslant x \leqslant \xi + \delta$, and $n_0(\xi, \epsilon, N)$ so that $n_0 > N$ and $|r_{n_0}(\xi)| < \epsilon$. And, when n_0 has thus been fixed, we can choose $\delta_1(\xi, \epsilon, n_0) = \delta_1(\xi, \epsilon, N)$ so that $\delta_1 < \delta$ and $|s_{n_0}(x) - s_{n_0}(\xi)| < \epsilon$ for $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$. Thus $|r_n(x)| < 3\epsilon$ for $n = n_0 > N$ and $\xi - \delta_1 \leqslant x \leqslant \xi + \delta_1$, so that the series is quasi-uniformly convergent for $x = \xi$.

6. If a series is uniformly convergent at every point ξ of an interval, it is (as we saw in § 4) uniformly convergent throughout the interval: definition **A 3** (and *a fortiori* definition **A 2**) passes over, in virtue of the Heine-Borel Theorem, into definition **A 1**. It is important to observe that this relation does not hold between **B 3** (or **B 2**) and **B 1**: a series quasi-uniformly convergent at every point of an interval (or in the neighbourhood of every such point) is not necessarily quasi-uniformly convergent throughout the interval. We can apply the Heine-Borel Theorem in the manner indicated in the first sentences of the footnote * to p. 152; but the last stage of the argument, in which every one of a finite number of different integers is replaced by the largest of them, fails. What we obtain is *the necessary and sufficient condition that $s(x)$ should be continuous throughout the interval*; and this is not

* *Fondamenti...*, p. 107 (German translation, *Grundlagen...*, pp. 143–145).

the condition **B 1** but a condition first formulated by Arzelà*, viz.:

C: Quasi-uniform convergence by intervals (*convergenza uniforme a tratti*). *The series is said to be quasi-uniformly convergent by intervals if to every positive* ϵ *and every* N *correspond a division of* (a, b) *into a finite number* $\nu(\epsilon, N)$ *of intervals* $\delta_r(\epsilon, N)$, *and a corresponding number of numbers* $n_r(\epsilon, N)$, *all greater than* N, *and such that* (A) *is true for* $n = n_r\ (r = 1, 2, \ldots, \nu)$ *and all values of* x *which belong to* δ_r.

The deduction of Arzelà's criterion from **B 3**, in the manner sketched above, was first made by Hobson†.

There is one further point which seems worth noticing here, although it is not directly connected with Stokes's memoir. Dini‡ proved that *if* $u_n(x) \geqslant 0$ *for all values of* n *and* x, *and* $s(x)$ *is continuous throughout* (a, b), *then the series is uniformly convergent throughout* (a, b). This theorem is now almost intuitive. For it is obvious that, for series of positive terms, quasi-uniform convergence in any one of the senses **B 1**, **B 2**, or **B 3** involves uniform convergence in the corresponding sense **A 1**, **A 2**, or **A 3**. If then $s(x)$ is continuous throughout (a, b) it is continuous for every ξ of (a, b); and therefore the series is quasi-uniformly convergent for every ξ; and therefore uniformly convergent for every ξ; and therefore uniformly convergent throughout (a, b).

7. Let us now consider Stokes's definitions and proofs in the light of the preceding discussion.

It is clear, in the first place, that Stokes has in his mind some phenomenon characteristic of *a small, but fixed, neighbourhood of a point.*

'Let $u_1 + u_2 + \ldots$ (66)', he says§, 'be a convergent infinite series having U for its sum. Let $v_1 + v_2 + \ldots$ (67) be another infinite series of which the general term v_n is a function of the positive variable h and becomes equal to u_n when h vanishes. Suppose that *for a sufficiently small value of* h *and all inferior values* the series (67) is convergent, and has V for its sum. It might at first sight be supposed that the limit of V for $h = 0$ was necessarily equal to U. This however is not true....

'THEOREM. The limit of V can never differ from U unless the convergency of the series (67) becomes infinitely slow when h vanishes.

* 'Sulle serie di funzioni', *Memorie di Bologna*, ser. 5, vol. 8, 1900, pp. 131–186, 701–744.

† *L. c.*, pp. 380–382.

‡ *L. c.* (German edition), pp. 148–149. See also Bromwich, *Infinite series*, p. 125 (Ex. 3).

§ p. 279.

'The convergency of the series is here said to become infinitely slow when, if n be the number of terms which must be taken in order to render the sum of the neglected series numerically less than a given quantity e, which may be as small as we please, n increases beyond all limit as h decreases beyond all limit.

'DEMONSTRATION. If the convergency do not become infinitely slow it will be possible to find *a number n, so great that for the value of h we begin with and for all inferior values greater than zero the sum of the neglected terms shall be numerically less than e....*'

Stokes's words, and in particular those which I have italicised, seem to me to make two things perfectly clear.

(1) Stokes is considering neither a property of an interval (a, b) *im Grossen* (such as is contemplated in **A 1** or **B 1**), nor a property of a single point which (as in **A 3** or **B 3**) need not be shared by any neighbouring point, but a property of an interval *im Kleinen*, that is to say *a small but fixed interval chosen to include a particular point.* His definition is therefore one of the type of **A 2** or **B 2**.

Stokes's failure to perceive the bearing of his discovery on problems of integration is made much more natural when we realise that he is considering throughout a neighbourhood of a point and not an interval *im Grossen.* And this remark applies to Seidel as well.

(2) Stokes is considering an inequality satisfied for a special value of n, or at most an infinite sequence of values of n, and *not* necessarily for all values of n from a certain point onwards. In this respect there is a quite sharp distinction between Stokes's work and Seidel's. What Stokes defines is (to use the language of this note) a mode of *quasi-uniform* convergence and not one of strictly uniform convergence.

It seems to me, then, that what Stokes defines is what I have called *quasi-uniform convergence in the neighbourhood of a point* (**B 2**).

8. If we adopt this view, Stokes's mistake becomes very much more intelligible. He proves, quite correctly, that uniform convergence in his sense implies continuity: his proof, stated quite formally and by means of inequalities, is substantially that given in § 5, under (1). He then continues* as follows.

'Conversely, if (66) is convergent, and if $U = V_0$†, the convergency of the series (67) cannot become infinitely slow when h

* p. 282. The italics are mine.

† V_0 is what Stokes calls 'the value of V for $h=0$', by which he means, of course, its limit when h tends to 0.

vanishes. For if U_n', V_n' represent the sums of the terms after the nth in the series (66), (67) respectively, we have

$$V = V_n + V_n', \; U = U_n + U_n';$$

whence

$$V_n' = V - U - (V_n - U_n) + U_n'.$$

Now $V - U$, $V_n - U_n$ vanish with h, and U_n' vanishes when n becomes infinite. Hence *for a sufficiently small value of h and all inferior values, together with a value of n sufficiently large and independent of h*, the value of V_n' may be made numerically less than any given quantity e however small; *and therefore, by definition, the convergency of the series* (67) *does not become infinitely slow when h vanishes.*'

Now this argument is, until we reach the last sentence, perfectly accurate, and indeed, if we translate it into inequalities, substantially identical with that given in § 5, under (2). Stokes proves, in fact, that continuity at ξ involves quasi-uniform convergence at ξ. Where he falls into error is simply in his final assertion that this property is that which he has previously defined, the mistake being due to a failure to observe that his intervals of values of h depend upon a prior choice of ϵ. In a word, he confuses, momentarily, **B 2** and **B 3**. The ordinary view that Stokes defined uniform convergence in the same sense as Weierstrass compels us to suppose that he confused **B 3** with **A 1**, or at any rate with **A 2**: and this is hardly credible.

I add one final remark. If we could identify Stokes's idea with **B 3**, instead of with **B 2**, we could acquit him of having made any mistake at all, since **B 3** really is a necessary and sufficient condition for continuity. We could then regard Stokes as having anticipated Dini's theorem. This view, however, does not seem to me to be tenable.